The Chi-Square (X^2) Statistic

- STAT-C-301 Sampling Distributions
- UNIT III

Section A

Use of the Chi-Square Statistic in a Test of Association Between a Risk Factor and a Disease

The Chi-Square (X^2) Statistic

- Categorical data may be displayed in **contingency tables**
- The **chi-square statistic** compares the observed count in each table cell to the count which would be expected **under the assumption of no association** between the row and column classifications
- The chi-square statistic may be used to test the hypothesis of no association between two or more groups, populations, or criteria
- Observed counts are compared to expected counts

Test of Independence of Attributes

CONTINGENCY TABLES

Displaying Data in a Contingency Table

Criterion 2	Criterion 1					
	1	2	3	. . .	C	Total
1	n_{11}	n_{12}	n_{13}	. . .	n_{1c}	r_1
2	n_{21}	n_{22}	n_{23}	. . .	n_{2c}	r_2
3	n_{31}				. . .	
. .	. .				. . .	
r	n_{r1}	. . .	. . .	. . .	n_{rc}	r_r
Total	c_1	c_2			c_c	n

Chi-Square Test Statistic

- The test statistic is:

$$c^2 = \sum_{i=1}^{k} \left[\frac{(O_i - E_i)^2}{E_i} \right]$$

- The degrees of freedom are:
  - (r–1)(c–1)
  - r = # of rows and c = # of columns
- Where:
  - O_i = the observed frequency in the ith cell of the table
  - E_i = the expected frequency in the ith cell of the table

Example: Is Disease Associated With Exposure?

- The relationship between disease and exposure may be displayed in a contingency table
- We can see that:

  37/54 = 68 % of **diseased** individuals were exposed

  13/66 = 20 % of **non-diseased** were exposed
- Do these data suggest an association between disease and exposure?

Disease			
Exposure	Yes	No	Total
Yes	37	13	50
No	17	53	70
Total	54	66	120

Observed Counts

- The **observed numbers or counts** in the table are:

Disease			
Exposure	Yes	No	Total
Yes	37	13	50
No	17	53	70
Total	54	66	120

Test of No Association

- **Question of interest: is disease associated with exposure?**
- Calculate what numbers of “exposed” and “non-exposed” individuals would be **expected** in each disease group **if** the probability of disease were the same in both groups
- If there was **no association** between exposure and disease, then the expected counts should nearly equal the observed counts, and the value of the chi-square statistic would be small
- In this example, we can calculate:

  Overall proportion with exposure = 50/120 = 0.42

  Overall proportion without exposure = 70/120 = 0.58 = 1– 0.42

Expected Counts

- Under the assumption of no association between exposure and disease, the expected numbers or counts in the table are:

Disease			
Exposure	Yes	No	Total
Yes	50/120 x 54 = 22.5	50/120 x 66 = 27.5	50
No	70/120 x 54 = 31.5	70/120 x 66 = 38.5	70
Total	54	66	120

Chi-Squared (X^2) Statistic

$$\chi^2 = \sum_i \frac{(O_i - E_i)^2}{E_i}$$

$$= \frac{(37 - 22.5)^2}{22.5} + \frac{(13 - 27.5)^2}{27.5}$$

$$+ \frac{(17 - 31.5)^2}{31.5} + \frac{(53 - 38.5)^2}{38.5}$$

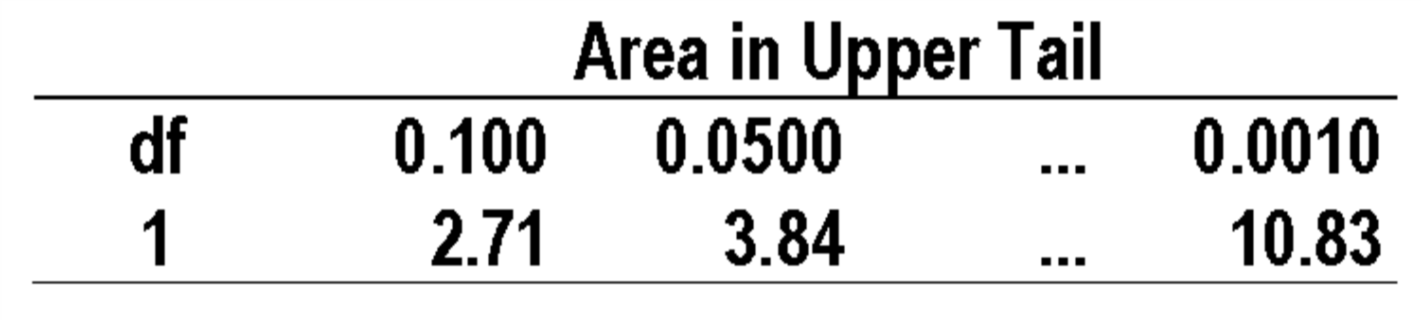
Test of Association

- The **test statistic** is:
  - $\chi^2 = 29.1$ with 1 degree of freedom
- **Assumption:** no association between disease and exposure
- A small value of the χ^2 statistic supports this assumption (observed counts and expected counts would be similar)
- A large value of the χ^2 statistic would not support this assumption (observed counts and expected counts would differ)
- What is the probability of obtaining a statistic of this magnitude or larger when there is no association?

Probability Associated with a X^2 Statistic

- If the assumption of no association is true, then what is the probability of observing this value of the χ^2 statistic?

- Table A.8 of the Pagano text provides the probability (area in upper tail of the distribution) associated with values of the chi-square statistic for varying degrees of freedom
- Degrees of freedom = 1 for a 2x2 table:



	Area in Upper Tail			
df	0.100	0.0500	...	0.0010
1	2.71	3.84	...	10.83

Conclusion: Test of (No) Association

- For the data in this example, $\chi^2 = 29.1$ with 1 degree of freedom
- From the chi-squared table, the probability obtaining a statistic of this magnitude or larger when there is no association is < 0.001
- In other words, the probability of obtaining discrepancies between observed and expected counts of this magnitude is < 0.001 (unlikely to occur by chance alone)
- Conclude that our finding is unlikely to occur if there is no association between disease and exposure
  - Thus, we conclude that there appears to be an association

Guidelines for Interpreting the X^2 Statistic

- The χ^2 statistic is calculated under the assumption of no association

- **Large value of χ^2 statistic** ⇒ small probability of occurring by chance alone ($p < 0.05$) ⇒ conclude that **association** exists between disease and exposure

- **Small value of χ^2 statistic** ⇒ large probability of occurring by chance alone ($p > 0.05$) ⇒ conclude that **no association** exists between disease and exposure

Short-Cut X^2 Formula for 2x2 Table

- Suppose:

Disease			
Exposure	Yes	No	Total
Yes	a	b	a+b
No	c	d	c+d
Total	a+c	b+d	n

- Then we can write:

$$\chi_1^2 = \frac{n(ad-bc)^2}{(a+c)(b+d)(a+b)(c+d)}$$

Short-Cut X^2 Formula for the Example

- Using this formula for the previous example gives

$$\chi_1^2 = \frac{120[(37)(53) - (13)(17)]^2}{54(66)(50)(70)}$$

$$= 29.1$$

Disease			
Exposure	Yes	No	Total
Yes	37	13	50
No	17	53	70
Total	54	66	120

  - Same as before!

Helpful Hints Regarding the Chi-Square Statistic

- The calculations use expected and observed counts or frequencies, not proportions

- The χ^2 short-cut formula applies only to 2x2 tables

- Probabilities are available from tables and computing packages

Review

- The χ^2 statistic provides a statistical test for ascertaining whether an association exists between disease and exposure

- A **large value of the χ^2 statistic** indicates that the observed data are unlikely under an assumption of no association between disease and exposure ⇒ **small probability (p-value)** ⇒ **association**

- A **small value of the χ^2 statistic** indicates that the observed data are likely under an assumption of no association between disease and exposure ⇒ **large probability (p-value)** ⇒ **no association**

Section B

Applications of the Chi-Square Statistic in Epidemiology

Applications of the X^2 Statistic in Epidemiology

- Cohort study (2 samples)
- Case-control study (2 samples)
- Matched case-control study (paired cases and controls)

Cohort Study 2x2 Table

Disease			
Factor	Present (D)	Absent ($\bar{D}$)	Total
Present (F)	a	b	a+b
Absent ($\bar{F}$)	c	d	c+d
Total	a+c	b+d	N

Cohort Study: Measure of Association

- Assumptions:

  The two samples are independent

  - Let a+b = number of people **exposed** to the risk factor
  - Let c+d = number of people **not exposed** to the risk factor

- Assess whether there is association between exposure and disease by calculating the relative risk (RR)

Cohort Study: Relative Risk

- We can define the relative risk of disease:

$$p_1 = P(\text{disease} \mid \text{factor present}) = P(D \mid F)$$

$$p_2 = P(\text{disease} \mid \text{factor absent}) = P(D \mid \overline{F})$$

$$RR = \frac{p_1}{p_2}$$

$$= \frac{P(D \mid F)}{P(D \mid \overline{F})}$$ is called the **relative risk**

Cohort Study: Test of Association

- For these samples, we can estimate the relative risk as:

$$RR = \frac{\frac{a}{a+b}}{\frac{c}{c+d}}$$

- We can test the hypothesis that RR=1 by calculating the chi-square test statistic

$$\chi_1^2 = \frac{n(ad-bc)^2}{(a+c)(b+d)(a+b)(c+d)}$$

Example: Test of Association in a Cohort Study

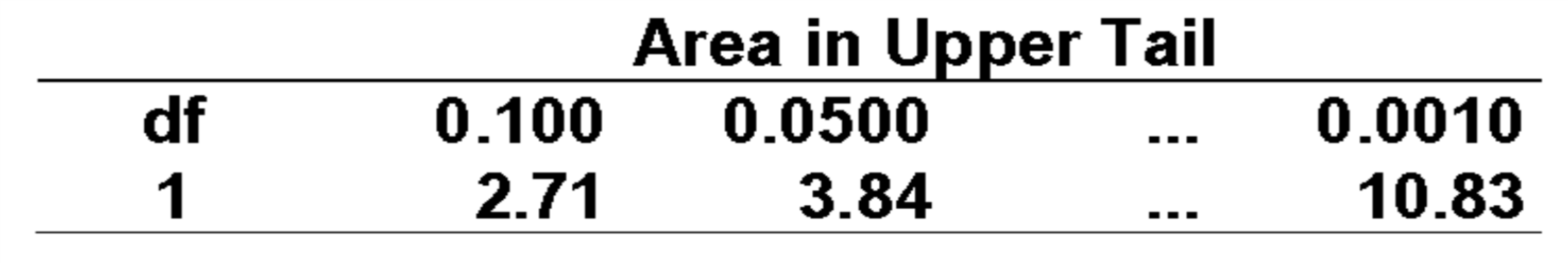
	Develop CHD		
Smoke	Yes	No	Total
Yes	84	2916	3000
No	87	4913	5000
Total	171	7829	8000

- RR = 1.61
- Chi-square statistic = χ^2_1 = 10.1 =

$$\frac{8000(84(4913)-2916(87))^2}{(84+87)(2916+4913)(84+2916)(87+4913)}$$

Example: Test of Association in a Cohort Study

- Using Table A.8 in the Pagano text, the probability is less than 0.010 (between 0.001 and 0.010)
- This supports an association between exposure and disease



	Area in Upper Tail			
df	0.100	0.0500	...	0.0010
1	2.71	3.84	...	10.83

Case-Control Study 2x2 Table

Disease			
Factor	Present (case)	Absent (control)	Total
Present	a	b	a+b
Absent	c	d	c+d
Total	a+c	b+d	N

Case-Control Study: Measure of Association

- Assumptions
  - The samples are independent
    - **Cases** = diseased individuals = a+c
    - **Controls** = non-diseased individuals = **b+d**

- We are interested in whether:

$$P(F|D) = P(F|\overline{D})$$

- We cannot estimate P(D), the prevalence of the disease and, hence, cannot estimate the RR

- Assess whether there is association between exposure and disease by calculating the odds ratio (OR)

Case-Control Study: Odds for Diseased Group

- The **odds of exposure for the diseased group** is:

$$\frac{p_1}{1-p_1} = \frac{\frac{a}{a+c}}{\frac{c}{a+c}} = \frac{a}{c}$$

	Disease		
Factor	Present (case)	Absent (control)	Total
Present	a	b	a+b
Absent	c	d	c+d
Total	a+c	b+d	N

Case-Control Study: Odds for Non-Diseased Group

- The **odds of exposure for the non-diseased group** is:

$$\frac{p_2}{1-p_2} = \frac{\frac{b}{b+d}}{\frac{d}{b+d}} = \frac{b}{d}$$

Disease			
Factor	Present (case)	Absent (control)	Total
Present	a	b	a+b
Absent	c	d	c+d
Total	a+c	b+d	N

Case-Control Study: Odds Ratio

- The odds ratio is:

$$\frac{\frac{p_1}{1-p_1}}{\frac{p_2}{1-p_2}}$$

- And is estimated by OR=

$$\frac{\frac{a}{c}}{\frac{b}{d}} = \frac{ad}{bc}$$

Case-Control Study: Test of Association

- We can test whether OR=1 by calculating the chi-square statistic:

$$\chi_1^2 = \frac{n(ad - bc)^2}{(a+c)(b+d)(a+b)(c+d)}$$

Example: Test of Association in a Case-Control Study

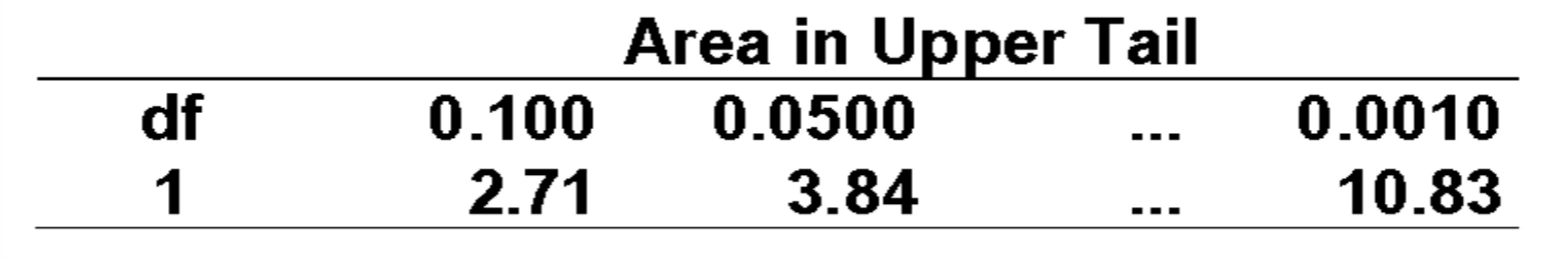
Disease			
Past Smoking	CHD Cases	Controls	Total
Yes	112	176	288
No	88	224	312
Total	200	400	600

- OR = 1.62
- Chi-square statistic = $\chi^2{}_1 = 7.69 =$

$$\frac{600(112(224)-176(88))^2}{(112+88)(176+224)(112+176)(88+224)}$$

Example: Test of Association in a Case-Control Study

- Using Table A.8 in the Pagano text, the probability is between 0.001 and 0.01
- This supports association between exposure and disease



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Matched Case-Control Study Table

		Controls	
		Exposed	Not Exposed
Cases	Exposed	aa	bb
	Not exposed	cc	dd

Matched Study: Measure of Association

- Assumptions
  - Case-control pairs are matched on characteristics such as age, race, sex
  - Samples are not independent

- The discordant pairs are case-control pairs with different exposure histories
  - The matched odds ratio is estimated by bb/cc
    - **Pairs in which cases exposed but controls not = bb**
    - **Pairs in which controls exposed but cases not = cc**
  - Assess whether there is association between exposure and disease by calculating the matched odds ratio (OR)

Matched Case-Control Study: Test of Association

- We can test whether OR = 1 by calculating McNemar's statistic

- McNemar's test statistic:

$$\chi_1^2 = \frac{(|bb - cc| - 1)^2}{(bb + cc)}$$

Example: Test of Association in a Matched Study

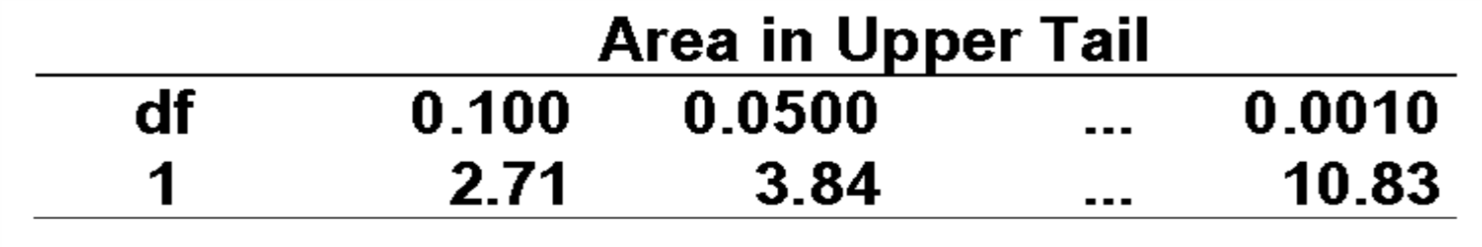
		Controls	
		Exposed	Not Exposed
Cases	Exposed	2	4
	Not exposed	1	3

- OR = bb/cc = 4
- McNemar's test statistic $=\chi^2_1 = 0.80$

$$\chi_1^2 = \frac{(|4-1|-1)^2}{(4+1)}$$

Example: Test of Association in a Matched Study

- Using Table A.8 in the Pagano text, the probability is greater than 0.100
- This supports no association between exposure and disease



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Review: Uses of the Chi-Squared Statistic

- The **chi-squared statistic** provides a test of the association between two or more groups, populations, or criteria
- **The chi-square test** can be used to test the strength of the association between exposure and disease in a cohort study, an unmatched case-control study, or a cross-sectional study
- **McNemar's test** can be used to test the strength of the association between exposure and disease in a matched case-control study