Question Bank

B.Sc(H) Mathematics-IV Sem

Riemann Integration and Series of Functions

Q1. Define $f(x) = [x]$ on [0,3]. Show that f is integrable and evaluate $\int_0^3 f(x)dx$.

Q2. Find the radius of convergence of the series $\sum_{n=2}^{\infty} \frac{1}{\ln n}$.

Q3. Suppose f is continuous function on [a,b], $f(x) \geq 0 \; \forall \; x \in [a, b]$. Then show that if $\int_a^b f(x)dx = 0$, then $f(x) = 0 \; \forall \; x \in [a, b]$.

Q4. Let f and g be continuous functions on [a,b] such that $\int_a^b f = \int_a^b g$. Prove that there exist $x \in [a, b]$ such that $f(x) = g(x)$.

Q5. (a) Give an example of a function f on [0,1] which is not integrable but $|f|$ is integrable.

(b)Show that if f is continuous real-valued function on [a,b] such that $\int_a^b f \,.\, g = 0$ for every continuous function g on [a,b] then show that $f(x) = 0$, for all $x \in [a, b]$.

Q6. Let $f \geq 0$ be integrable function on [a,b] . Is $\sqrt{f}$ integrable on $[a, b]$?

Q7. Let $f(x) = \sin\frac{1}{x}$ for $x \neq 0$ and $(0) = 0$. Show f is integrable on [-1, 1].

Q8. Let (f_n) be a sequence of integrable functions on $[a, b]$, and suppose $f_n \to f$ uniformly on $[a, b]$. Prove that f is integrable on $[a, b]$ and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

Q9. Show that if $a > 0$, then (f_n) defined as $f_n(x) = tan^{-1}(nx)$ converges uniformly to $f(x) = \frac{\pi}{2}\, sgn(x)$ on the interval $[a, \infty)$ but is not uniformly convergent on $(0, \infty)$.

Q10. Prove that $\int_\pi^\infty \frac{\sin x}{x}\, dx$ converges absolutely.

Q11. Let $f_n(x) = \frac{1}{(1+x)^n}$ for $x \in [0,1]$. Find the pointwise limit f of the sequence (f_n) on [0,1]. Does (f_n) converges uniformly to f on [0,1]?

Q12. Suppose a sequence (f_n) converges uniformly to f on the set A, and suppose that each f_n is bounded on A. Show that the function f is bounded on A.

Q13. Prove that lim sup ($|na_n|^{\frac{1}{n}}$) = lim sup ($|a_n|^{\frac{1}{n}}$).

Q14. Let $f(x) = \sum a_n x^n$ for $|x| < R$. If $f(x) = f(-x)$ for all $|x| < R$, show that $a_n = 0$ for all odd n.

Q15. (a)Give an example of an integrable function which has an infinite set of points of discontinuity having only one limit point.

(b)Give an example of a Riemann integrable function which is not monotonic.

Q16. Show that every continuous function on [a,b] is integrable. Is the converse true? Justify.

Q17. Define g:[0,2]→ [0,4] by g(x)=x^2 and let P=$\{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$. Find U(g,P).

Q18. Prove that the bounded function f is integrable iff for each ϵ>0 there exist δ>0 such that U(f,P)-L(f,P)< ϵ whenever mesh(P)< δ, ∀ partitions P of [a,b].

Q19. Is the sequence $< f_n = \frac{1}{n} sin(nx + n) >$ uniformly convergent on R? Justify your answer.

Q20. Suppose a sequence $< f_n >$ converges uniformly to f on a set A such that each f_n bounded on A. Show that limit function f is bounded on A.

Q21. Show that if $0 < b < 1$, then the convergence of the sequence $f_n = \frac{x^n}{1+x^n}$, for x in R, $x \geq 0$ is uniform on the interval $[0, b]$ but not uniform on the interval [0,1].

Q22. Let $f_n(x) = \frac{1}{(1+x)^n}$ for $x \in$ [0,1], n ∈N. Find the pointwise limit f of the sequence $< f_n >$ on [0,1]. Does $< f_n >$ converge uniformly to f on [0,1]?

Q23. Let $f_n(x) = \frac{\sin nx}{1+nx}$ for $x \geq 0$. Show that the sequence $< f_n >$ converges only pointwise on [0,∞) and converges uniformly on [a,∞), a> 0.