Unique Paper Code	: **32351501**
Name of Course	: **B.Sc. (H) Mathematics**
Name of Paper	: **C11-Metric Spaces**
Semester	: **V**
Duration	: **3 Hours**
Maximum Marks	: **75**

Attempt any four questions. All questions carry equal marks.

1. Prove that a metric space (X, d) is compact if and only if every collection of closed sets in X with empty intersection has a finite subcollection with empty intersection.

Do there exist onto continuous functions

$$f_1: [0,1] \to \mathbb{Q}$$
$$f_2: [0,1] \to [2,3] \cup [4,5]$$
$$f_3: [0,1] \to (5,7)$$
$$f_4: [0,1] \to [2,4]$$
$$f_5: [0,1] \to (10, +\infty)$$
$$f_6: [0,1] \to (0,1]$$
$$f_7: (1, \infty) \to (0,1).$$

If yes, give the explicit expression for the function. If no, then clearly state the reason. **(4.75+14)**

2. Let $X = \mathbb{R}$, d_1 be the usual metric and d_2 be the discrete metric. Let $A = (0,1) \cup \{5\}$. Find the interior, derived set, closure and diameter of A in the metric spaces (X, d_1) and (X, d_2). Also, find the distance between the point 7 and the set A in the metric spaces (X, d_1) and (X, d_2).

Give an example of an open dense subset of (X, d_1) which is uncountable. Are the metric spaces (X, d_1) and (X, d_2) separable? Justify.

Prove that the function $f: (X, d_2) \to (X, d_1)$ defined by $f(x) = x$ is continuous, one-one and onto, but not a homeomorphism. **(8+2+1+4+3.75)**

3. Let $X = \mathbb{R}^2$ with Euclidean metric d. Prove that (X, d) is a complete metric space.

Verify the Cantor Intersection Theorem for the sequence (F_n) of subsets of X, where $F_n = \bar{S}((0,0), 1/n)$ where $\bar{S}(x, r)$ denotes the closed ball centred at x and radius r.

Determine which of the following subsets of $(\mathbb{R}^2, d)$ are complete? Justify your answer in each case.

$$A_1 = \{(x, y): y = x\}$$
$$A_2 = \{(x, y): x > 0\}$$
$$A_3 = \{(x, y): x = 0\} \cup \{(x, y): y = 0\}$$
$$A_4 = \{(x, y): 1 < y < 2\}.$$

Prove that the metrics d and d_∞ are equivalent on $\mathbb{R}^2$, where $d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$ for $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Is (X, d_∞) a complete metric space? Justify. **(4+4+4+4+2.75)**

4. Let $X = \mathbb{R}$ with usual metric and $Y = [0,1)$. Show that the subsets $F_1 = [0,0.2]$ and $F_2 = [0.8,1)$ are closed in Y. Also, show that the subsets $G_1 = (0.1,0.2)$ and $[0,0.4)$ are open in Y. Find the interior and closure of the subsets $(0,0.5), (0.5,1)$ and $[0,1)$ in Y. Prove that Y is a separable metric space. Is Y complete? Is Y connected? Is Y compact? Justify your answer in each case. **(2+2+6+2.75+2+2+2)**

5. If (X, d_X) is a disconnected metric space, prove that there exists a continuous function $f: (X, d_X) \to (\mathbb{R}, d)$ which does not have Intermediate Value Property. Hence or otherwise prove that every continuous function f on the interval $[-1, 1]$ with $|f(x)| \leq 1 \ \forall x \in [-1, 1]$ has a fixed point.

What can be said about the connectedness of $\mathbb{Q}$, w.r.t. the metric

$$d(x, y) = \frac{|x-y|}{1+|x-y|}?$$

Justify.

For a subset Y of $\mathbb{R}$ such that $(0, 1) \subset Y \subset [0, 1]$, what can be concluded about its connectedness and why ?

Is the diameter of a connected set zero? Justify your answer by an example.

Give an example of an infinite connected set with finite diameter. **(5+4+4+2+1.75+2)**

6. For a metric space (X, d) and a continuous function f from X into itself, show that the set of points $\{x: f(x) = x\}$ is a closed subset of X.

If f is a continuous function from a compact metric space (X, d_X) into an arbitrary metric space (Y, d_Y), prove that f is uniformly continuous and the image of X under f is compact.

Further, in addition if (X, d_X) and (Y, d_Y) both are homeomorphic, what can be concluded about Y? Justify.

Hence prove that there exist $x_1, x_2 \in [a, b]$ such that $f(x_1) = \sup_{x \in [a,b]} f(x)$ and $f(x_2) = \inf_{x \in [a,b]} f(x)$, where $f: [a, b] \to \mathbb{R}$ is such that $f(t) = \alpha t + \beta$, for some α and β in $\mathbb{R}$.

(5+6+3+3.75)