Question Bank

B.Sc. (h) Mathematics-V Semester

Metric Spaces

Q1. Is A, B are dense in X, is A ∪ B dense in X? Is A ∩ B dense in X?

Q2. (a) Prove that the metric spaces, R with the usual metric and $(0, \infty)$ with the usual metric induced from R are homeomorphic.
(b) Show that (0,1] and [1, ∞) are homeomorphic under usual metric.

Q3. Is the set A = $\{(x, y) : x + y = 1\}$ open in the metric space (R^2, d_2)? Justify.

Q4. Let F be a subset of a metric space (X, d). Prove that the set of limit points of F is closed subset of (X, d).

Q5. Let F be a non-empty bounded closed subset of R, with usual metric and $a = sup$ F. Prove that $a \in$ F.

Q6. Let (X, d) be any metric space and $f : (X, d) \to (R^n, d_2)$ be defined by: $f(x) = \big(f_1(x), f_2(x), \dots, f_n(x)\big)$, for $x \in X$. Show that if f is continuous, so is each $f_k : X \to R$, $k = 1, 2, \dots, n$.

Q7. Give an example of complete metric space X and a function $f : X \to X$ such that

$d\big(f(x), f(y)\big) \leq d(x, y)$ for all $x, y \in X$, but f has no fixed point in X.

Q8. If $x = < x_n >$ has a subsequence which converges to z. Show that

dist(z,$\{ x_n : n \in N\}$) = 0.

Give an example to show that converse of above statement is not necessarily true.

Q9. Suppose X is a metric space and S is non-empty subset of X. Then prove that diam$\big(cl(S)\big)$ = diam(S). Do also we have diam(S°) = diam(S)? Justify your answer.

Q10. Suppose X is a metric space and $S \subseteq X$. Prove that
(a) $(S^\circ)^c = cl(S^c)$ (b) $S^\circ = \{x \in X :$ dist$(x, S^c) > 0\}$.

Q11. Suppose (X, d) is a metric space, $x \in X$ and A is a subset of X. Then show that dist(x, cl(A)) = dist(x, A), where cl(A) is the closure of A.

Q12. Let (X, d) be a metric space and $Y \subseteq X$ be connected. If $Y \subseteq A \cup$ B where A and B are separated sets in X then prove that either $Y \subseteq$ A or $Y \subseteq$ B.

Q13. Can a metric space be empty? What is the use of metric space in real life?

Q14. Why is every metric space open?

Q15. Let d be a metric on X. Determine all constants k such that (i) kd , (ii) d + k

is a metric on X.

Q16. Describe the closure of each of the following Subsets:

(a) The integers on R. (b) The rational numbers on R.

(c) The complex number with real and imaginary parts as rational in C.

(d) The disk $\{z : |z| < 1\} \subset C$.

Q17. If (x_n) and (y_n) are Cauchy sequences in a metric space (X, d) show that (a_n), where $a_n = d(x_n, y_n)$, converges.

Q18. Let X=R and for x, y ∈ R, define d(x, y) by

$$d(x,y) = \begin{cases} |x-y|+1, & \text{if exactly one of } x \text{ and } y \text{ is strictly positive} \\ |x-y|, & \text{otherwise} \end{cases}$$

Prove that (X, d) is a metric space.

Q19. Prove that (0,1) with absolute value metric is not complete but (0,1) with discrete metric is complete.