**Question Bank**
B.Sc(H) Mathematics-III Semester
Group Theory-I

Q1. Prove that factor group of cyclic/abelian group is cyclic/abelian.

Q2. If H is a normal subgroup of a group G, prove that C(H), the centralizer of H in G, is a normal subgroup of G.

Q3. Let *G* be a group and *p* a prime. Suppose that $H = \{g^p : g \in G\}$ is a subgroup of *G*. Show that *H* is normal and that every nonidentity element of $G/H$ has order *p*.

Q4. Write out a complete Cayley table for $D_3$. Is $D_3$ Abelian?

Q5. Find elements A, B, and C in $D_4$ such that AB = BC but A ≠ C.

Q6. In each case, find the inverse of the element under the given operation.
a) 13 in $Z_{20}$ b) 13 in U(14) c) n-1 in U(n) (n>2)
d) 3-2i in C*, the group of nonzero complex numbers under multiplication

Q7. For any elements a and b from a group and any integer n, prove that $(a^{-1}ba)^n = a^{-1}b^na$.

Q8. List the six elements of GL(2, $Z_2$). Show that this group is non Abelian by finding two elements that do not commute.

Q9. Let *G* be a group. Show that $Z(G) = \cap_{a \in G} C(a)$.

Q10. Let *G* be a group, and let $a \in G$. Prove that $C(a) = C(a^{-1})$.

Q11. Prove that $S_n$ is non-Abelian for all $n \geq 3$. Also, Show that for $n \geq 3$, $Z(S_n) = \{e\}$.

Q12. Let N be a normal subgroup of G and let H be a subgroup of G. If N is a subgroup of H, prove that H/N is a normal subgroup of G/N if and only if H is a normal subgroup of G.

Q13. Show that the intersection of two normal subgroups of G is a normal subgroup of G. Generalize.

Q14. If *H* is a normal subgroup of *G* and $|H| = 2$, prove that *H* is contained in the center of *G*.

Q15. Describe the symmetries of a nonsquare rectangle. Construct the corresponding Cayley table.

Q16. Show that the group GL(2, R) is non-Abelian. Also, find the inverse of the element $\begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}$ in GL(2, $Z_{11}$).

Q17. List the members of $H = \{x^2 \mid x \in D_4\}$ and $K = \{x \in D_4 \mid x^2 = e\}$.

Q18. Let G be a finite group. Show that the number of elements x of G such that $x^3 = e$ is odd. Show that the number of elements x of G such that $x^2 \neq e$ is even.

Q19. Let G ={$\begin{bmatrix} a & a \\ a & a \end{bmatrix}$ : a ∈ R, a≠0}. Show that G is a group under matrix multiplication. Explain why each element of G has an inverse even though the matrices have 0 determinants.

Q20. Suppose *G* is a group that has exactly eight elements of order 3. How many subgroups of order 3 does *G* have?

Q21. Let G = GL(2, R). a) Find C ($\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$) b) Find C($\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$) c) Find Z(G).

Q22. Let Z denote the group of integers under addition. Is every subgroup of Z cyclic? Why? Describe all the subgroups of Z. Let a be a group element with infinite order. Describe all subgroups of <a>.

Q23. Give an example of subgroups H and K of a group G such that HK is not a subgroup of G.

Q24. If N and M are normal subgroups of G, prove that NM is also a normal subgroup of G.

Q25. Let N be a normal subgroup of a group G. If N is cyclic, prove that every subgroup of N is also normal in G.

Q26. Show that {1, 2, 3} under multiplication modulo 4 is not a group but that {1, 2, 3, 4} under multiplication modulo 5 is a group

Q27. Give an example of a group with 105 elements. Give two examples of groups with 44 elements.

Q28. Prove that the set of all rational numbers of the form $3^m6^n$, where m and n are integers, is a group under multiplication. Also, let m and n be elements of the group (Z,+). Find a generator for the group <m ∩ n>.

Q29. For any elements a and b from a group and any integer n, prove that
a) $(a^{-1}ba)^n = a^{-1}b^na$, b) $|ab| = |ba|$, c) $|ab| = |a^{-1}b^{-1}|$.

Q30. Let H = {A∈GL(2,R) | det A is an integer power of 2}. Show that H is a subgroup of GL(2, R).

Q31. Let G = GL(2, R) and H ={ $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ : a and b are nonzero integers} under the operation of matrix multiplication. Prove or disprove that H is a subgroup of GL(2, R).

Q32. Show that the group of positive rational numbers under multiplication is not cyclic.